

THE MINIMAL FREE RESOLUTION OF A STAR-CONFIGURATION IN \mathbb{P}^n

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ABSTRACT. We find the minimal free resolution of the ideal of a star-configuration in \mathbb{P}^n of type (r, s) defined by general forms in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$. This generalises the results of [2, 9] from a specific value of $r = 2$ to any value of $1 \leq r \leq n$. Moreover, we show that any star-configuration in \mathbb{P}^n is arithmetically Cohen-Macaulay. As an application, we construct a few of graded Artinian rings, which have the weak Lefschetz property, using the sum of two ideals of star-configurations in \mathbb{P}^n .

1. INTRODUCTION

Let $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ be an $(n + 1)$ -variable polynomial ring over an infinite field \mathbb{k} of any characteristic, and I a homogeneous ideal of R (or the ideal of a subscheme in \mathbb{P}^n). Then the numerical function

$$\mathbf{H}(R/I, t) := \dim_{\mathbb{k}} R_t - \dim_{\mathbb{k}} I_t$$

is called a *Hilbert function* of the ring R/I . If $I := I_{\mathbb{X}}$ is the ideal of a subscheme \mathbb{X} in \mathbb{P}^n , then we denote the Hilbert function of \mathbb{X} by

$$\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}(R/I_{\mathbb{X}}, t).$$

A star-configuration has been studied to calculate the dimension of the secant variety to a variety of reducible forms in R . In [14], the authors first introduced it as extremal points sets with maximal Hilbert function and as the support of a family of $\binom{s}{2}$ fat points. Other applications of such star-configurations have been studied in the work of [4, 6, 7, 9, 17, 18]. In [3], the authors proved that if F_1, \dots, F_s are general forms in R and

$$\tilde{F}_j = \frac{\prod_{i=1}^s F_i}{F_j} \quad \text{for } j = 1, \dots, s,$$

then

$$\bigcap_{1 \leq i < j \leq s} (F_i, F_j) = (\tilde{F}_1, \dots, \tilde{F}_s).$$

They called the variety \mathbb{X} in \mathbb{P}^n of the ideal $(\tilde{F}_1, \dots, \tilde{F}_s)$ a *star-configuration* in \mathbb{P}^n of type $(2, s)$ defined by general forms F_1, \dots, F_s . In this paper, we generalise the definition of a star-configuration in \mathbb{P}^n as follows: Let F_1, \dots, F_s be general forms in R and let $1 \leq r \leq \min\{s, n\}$. Then the variety \mathbb{X} of the ideal

$$\bigcap_{1 \leq i_1 \leq \dots \leq i_r \leq s} (F_{i_1}, \dots, F_{i_r})$$

is called a *star-configuration* in \mathbb{P}^n of type (r, s) defined by general forms F_1, \dots, F_s . Furthermore, if F_1, \dots, F_s are all general *linear* forms, then \mathbb{X} is called a *linear star-configuration* in \mathbb{P}^n of type (r, s) (see also [2, 3, 17, 18]).

Recently, the minimal free resolution for the ideal $I_{\mathbb{X}}$ of some specific star-configurations were found. More precisely, in [3], the authors found the minimal free resolution of the ideal of a star-configuration in

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\mathbb{P}^n of type $(2, s)$. In [9], the authors found the minimal free resolutions of the ideal $I_{\mathbb{X}}$ of a linear star-configuration \mathbb{X} in \mathbb{P}^n of type (r, s) and the 2-nd symbolic power of $I_{\mathbb{X}}$. Moreover, they showed that a linear star-configuration \mathbb{X} in \mathbb{P}^n is arithmetically Cohen-Macaulay (*aCM* for short).

In Section 2 we find the minimal generators of the ideal $I_{\mathbb{X}}$ of a star-configuration \mathbb{X} in \mathbb{P}^n of type (r, s) defined by s general forms in R . In Section 3 we find the minimal free resolution of the ideal $I_{\mathbb{X}}$ using Eagon-Northcott resolution (see Theorem 3.6), which generalises the results of [2, 9]. We also prove that every star-configuration is aCM (see Theorem 3.6), which generalises the interesting result of [9].

As an application, we discuss the weak Lefschetz-property in Section 4. This fundamental property has been studied by many authors (see [1, 2, 5, 10, 11, 13, 15, 16, 18]). In [18], the author proved that if \mathbb{X} is the union of two star-configurations in \mathbb{P}^2 defined by linear forms and quadratic forms and $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property. In this section we also find another Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ having the weak Lefschetz property without the condition $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$ (see Proposition 4.11). We also prove that if \mathbb{X} and \mathbb{Y} are star-configurations in \mathbb{P}^2 of type $(2, s)$ and $(2, s + 1)$ defined by forms F_1, \dots, F_s , and G_1, \dots, G_s, L in $R = \mathbb{k}[x_0, x_1, x_2]$, respectively, with $\deg(F_i) = \deg(G_i) \leq 2$ for $i = 1, \dots, s$, and L is a general linear form in R , then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property with a Lefschetz element L (see Theorem 4.17), which generalises the result of [18].

2. THE MINIMAL GENERATORS OF THE IDEAL OF A STAR-CONFIGURATIONS IN \mathbb{P}^n

Definition-Remark 2.1. Let $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ be a polynomial ring over a field \mathbb{k} . For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose F_1, \dots, F_s are general forms in R of degrees d_1, \dots, d_s , respectively. We call the variety \mathbb{X} defined by the ideal

$$\bigcap_{1 \leq i_1 < \dots < i_r \leq s} (F_{i_1}, \dots, F_{i_r})$$

a star-configuration in \mathbb{P}^n of type (r, s) . In particular, if F_1, \dots, F_s are general linear forms in R , then we call \mathbb{X} a linear star-configuration in \mathbb{P}^n of type (r, s) .

Notice that each n -forms F_{i_1}, \dots, F_{i_n} define $d_{i_1} \cdots d_{i_n}$ points in \mathbb{P}^n for each $1 \leq i_1 < \dots < i_n \leq s$. Thus the ideal

$$\bigcap_{1 \leq i_1 < \dots < i_n \leq s} (F_{i_1}, \dots, F_{i_n})$$

defines a finite set \mathbb{X} of points in \mathbb{P}^n with

$$\deg(\mathbb{X}) = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq s} d_{i_1} d_{i_2} \cdots d_{i_n}.$$

Lemma 2.2. Let $F_1, \dots, F_s, G_1, \dots, G_t$ be general forms in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ with $s \geq 1$ and $n \geq 2$. Then

$$\bigcap_{1 \leq i \leq s} (F_i, G_1, \dots, G_t) = \left(\prod_{\ell=1}^s F_{\ell}, G_1, \dots, G_t \right)$$

for $0 \leq t \leq n - 1$.

Proof. We shall prove this by induction on s . If $s = 1$, then it is clear. So we assume that $s > 1$. Then, by induction on s ,

$$\begin{aligned} & \bigcap_{1 \leq i \leq s} (F_i, G_1, \dots, G_t) \\ &= \left[\bigcap_{1 \leq i \leq s-1} (F_i, G_1, \dots, G_t) \right] \cap (F_s, G_1, \dots, G_t) \\ &= \left(\prod_{\ell=1}^{s-1} F_{\ell}, G_1, \dots, G_t \right) \cap (F_s, G_1, \dots, G_t). \end{aligned}$$

First, it is obvious that

$$\left(\prod_{\ell=1}^s F_{\ell}, G_1, \dots, G_t \right) \subseteq \left(\prod_{\ell=1}^{s-1} F_{\ell}, G_1, \dots, G_t \right) \cap (F_s, G_1, \dots, G_t).$$

Conversely, assume that $F \in (\prod_{\ell=1}^{s-1} F_\ell, G_1, \dots, G_t) \cap (F_s, G_1, \dots, G_t)$. Then, for some $H_i, K_i, L_i, M_i \in R$,

$$F = (\prod_{\ell=1}^{s-1} F_\ell) H_1 + K_1 G_1 + \dots + K_t G_t = F_s M_s + L_1 G_1 + \dots + L_t G_t.$$

Moreover, since $n \geq t + 1$, the forms $\prod_{\ell=1}^{s-1} F_\ell, F_s, G_1, \dots, G_t$ are general forms in R . We thus get that $H_1 \in (F_s, G_1, \dots, G_t)$, and $F \in (\prod_{\ell=1}^s F_\ell, G_1, \dots, G_t)$, as we wished. \square

Theorem 2.3. *Let $F_1, \dots, F_s, G_1, \dots, G_t$ be general forms in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ for $s \geq 2, t \geq 0$, and $n \geq 2$. Then*

$$\bigcap_{1 \leq j_1 < \dots < j_r \leq s} (F_{j_1}, \dots, F_{j_r}, G_1, \dots, G_t) = \sum_{1 \leq i_1 < \dots < i_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} \dots F_{i_{r-1}}}, G_1, \dots, G_t \right)$$

for $1 \leq r \leq n - t$.

Proof. We shall prove this by double induction on r and s . If $r = 1$, then by Lemma 2.2, it holds for every $s \geq 1$. Now assume $r > 1$. If $s = 2$, then it is immediate. Let $s > 2$. By double induction on s and r ,

$$\begin{aligned} & \bigcap_{1 \leq j_1 < \dots < j_r \leq s} (F_{j_1}, \dots, F_{j_r}, G_1, \dots, G_t) \\ &= \left[\bigcap_{1 \leq j_1 < \dots < j_r \leq s-1} (F_{j_1}, \dots, F_{j_r}, G_1, \dots, G_t) \right] \cap \left[\bigcap_{1 \leq j_1 < \dots < j_{r-1} \leq s-1} (F_{j_1}, \dots, F_{j_{r-1}}, F_s, G_1, \dots, G_t) \right] \\ &= \left[\sum_{1 \leq i_1 < \dots < i_{r-1} \leq s-1} \left(\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-1}}}, G_1, \dots, G_t \right) \right] \cap \left[\sum_{1 \leq i_1 < \dots < i_{r-2} \leq s-1} \left(\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-2}}}, F_s, G_1, \dots, G_t \right) \right]. \end{aligned}$$

First, note that

$$\begin{aligned} & \sum_{1 \leq i_1 < \dots < i_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} \dots F_{i_{r-1}}}, G_1, \dots, G_t \right) \\ &\subseteq \left[\sum_{1 \leq i_1 < \dots < i_{r-1} \leq s-1} \left(\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-1}}}, G_1, \dots, G_t \right) \right] \cap \left[\sum_{1 \leq i_1 < \dots < i_{r-2} \leq s-1} \left(\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-2}}}, F_s, G_1, \dots, G_t \right) \right]. \end{aligned}$$

Conversely, assume that

$$F \in \left[\sum_{1 \leq i_1 < \dots < i_{r-1} \leq s-1} \left(\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-1}}}, G_1, \dots, G_t \right) \right] \cap \left[\sum_{1 \leq i_1 < \dots < i_{r-2} \leq s-1} \left(\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-2}}}, F_s, G_1, \dots, G_t \right) \right].$$

Then for some $K_{i_1 \dots i_{r-1}}, M_{j_1 \dots j_{r-2}}, H_i, L_i \in R$,

$$F = \sum_{1 \leq i_1 < \dots < i_{r-1} \leq s-1} \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} \dots F_{i_{r-1}}} \cdot K_{i_1 \dots i_{r-1}} + \sum_{i=1}^t H_i G_i \quad (2.1)$$

$$= \sum_{1 \leq j_1 < \dots < j_{r-2} \leq s-1} \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{j_1} \dots F_{j_{r-2}}} \cdot M_{j_1 \dots j_{r-2}} + F_s M_s + \sum_{i=1}^t L_i G_i. \quad (2.2)$$

Now we first show that, using two representation of F given in (2.1) and (2.2), $K_{i_1 \dots i_{r-1}}$ belongs to the ideal $(F_{i_1}, \dots, F_{i_{r-1}}, F_s, G_1, \dots, G_t)$. Indeed if $(i'_1, i'_2, \dots, i'_{r-1}) \neq (i_1, i_2, \dots, i_{r-1})$, then

$$\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i'_1} F_{i'_2} \dots F_{i'_{r-1}}} \in (F_{i_1}, F_{i_2}, \dots, F_{i_{r-1}}).$$

So,

$$\sum_{\substack{1 \leq i'_1 < i'_2 < \dots < i'_{r-1} \leq s-1 \\ (i'_1, i'_2, \dots, i'_{r-1}) \neq (i_1, i_2, \dots, i_{r-1})}} \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i'_1} F_{i'_2} \dots F_{i'_{r-1}}} \cdot K_{i'_1 i'_2 \dots i'_{r-1}} \in (F_{i_1}, F_{i_2}, \dots, F_{i_{r-1}}). \quad (2.3)$$

Moreover, we have that, for every $1 \leq j_1 < \dots < j_{r-2} \leq s-1$,

$$\{i_1, \dots, i_{r-1}\} \setminus \{j_1, \dots, j_{r-2}\} \neq \emptyset, \text{ i.e.,} \\ \sum_{1 \leq j_1 < \dots < j_{r-2} \leq s-1} \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{j_1} \dots F_{j_{r-2}}} \cdot M_{j_1 \dots j_{r-2}} \in (F_{i_1}, F_{i_2}, \dots, F_{i_{r-1}}).$$

It follows from the second representation of F in the equation (2.2) that

$$\begin{aligned} F &= \sum_{1 \leq j_1 < \dots < j_{r-2} \leq s-1} \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{j_1} \dots F_{j_{r-2}}} \cdot M_{j_1 \dots j_{r-2}} + F_s M_s + \sum_{i=1}^t L_i G_i. \\ &\in (F_{i_1}, F_{i_2}, \dots, F_{i_{r-1}}, F_s, G_1, \dots, G_t). \end{aligned} \quad (2.4)$$

Recall that $F_{i_1}, F_{i_2}, \dots, F_{i_{r-1}}, F_s, G_1, \dots, G_t$ are $(r+t)$ -general forms in R with $r+t \leq n$. Hence it follows from the equations (2.1), (2.3), and (2.4) that

$$K_{i_1 i_2 \dots i_{r-2} i_{r-1}} = F_{i_1} N_{i_1} + F_{i_2} N_{i_2} + \dots + F_{i_{r-1}} N_{i_{r-1}} + F_s N_s + \sum_{i=1}^t N'_i G_i$$

for some $N_{i_j}, N'_i \in R$. Thus we have

$$\begin{aligned} &\frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} F_{i_2} F_{i_3} \dots F_{i_{r-2}} F_{i_{r-1}}} \cdot K_{i_1 i_2 i_3 \dots i_{r-2} i_{r-1}} \\ &= \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} F_{i_2} F_{i_3} \dots F_{i_{r-2}} F_{i_{r-1}}} \cdot (F_{i_1} N_{i_1} + F_{i_2} N_{i_2} + F_{i_3} N_{i_3} + \dots + F_{i_{r-1}} N_{i_{r-1}} + F_s N_s + \sum_{i=1}^t N'_i G_i) \\ &= \frac{\prod_{\ell=1}^s F_\ell}{F_{i_2} F_{i_3} \dots F_{i_{r-2}} F_{i_{r-1}} F_s} \cdot N_{i_1} + \frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} F_{i_3} \dots F_{i_{r-2}} F_{i_{r-1}} F_s} \cdot N_{i_2} + \frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} F_{i_2} F_{i_4} \dots F_{i_{r-2}} F_{i_{r-1}} F_s} \cdot N_{i_3} + \dots + \\ &\quad \frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} F_{i_2} F_{i_3} \dots F_{i_{r-2}} F_s} \cdot N_{i_{r-1}} + \frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} F_{i_2} F_{i_3} \dots F_{i_{r-2}} F_{i_{r-1}}} \cdot N_s + \frac{\prod_{\ell=1}^{s-1} F_\ell}{F_{i_1} F_{i_2} F_{i_3} \dots F_{i_{r-2}} F_{i_{r-1}}} \cdot \sum_{i=1}^t N'_i G_i \\ &\in \sum_{1 \leq k_1 < \dots < k_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^s F_\ell}{F_{k_1} \dots F_{k_{r-1}}}, G_1, \dots, G_t \right). \end{aligned} \quad (2.5)$$

This holds for arbitrary chosen (i_1, \dots, i_{r-1}) with $1 \leq i_1 < \dots < i_{r-1} \leq s$. This implies that

$$F \in \sum_{1 \leq i_1 < \dots < i_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} \dots F_{i_{r-1}}}, G_1, \dots, G_t \right),$$

as we wished. This completes the proof. \square

The following corollary is immediate from Theorem 2.3 .

Corollary 2.4. *Let \mathbb{X} be a star-configuration in \mathbb{P}^n of type (r, s) defined by general forms F_1, \dots, F_s in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ with $1 \leq r \leq \min\{s, n\}$. Then*

$$I_{\mathbb{X}} = \bigcap_{1 \leq i_1 < \dots < i_r \leq s} (F_{i_1}, \dots, F_{i_r}) = \sum_{1 \leq j_1 < \dots < j_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^s F_\ell}{F_{j_1} \dots F_{j_{r-1}}} \right).$$

Corollary 2.5. *Let \mathbb{X} be a linear star-configuration in \mathbb{P}^n of type (n, s) defined by s general linear forms in $R = \mathbb{k}[x_0, \dots, x_n]$ with $s \geq n \geq 2$. Then \mathbb{X} has generic Hilbert function*

$$\mathbf{H}_{\mathbb{X}} : 1 \quad \binom{1+n}{n} \quad \dots \quad \binom{(s-n)+n}{n} \quad \binom{(s-n)+n}{n} \rightarrow,$$

i.e.,

$$\mathbf{H}_{\mathbb{X}}(i) = \min \left\{ \deg(\mathbb{X}), \binom{i+n}{n} \right\}$$

for every $i \geq 0$.

Proof. By Corollary 2.4, $I_{\mathbb{X}}$ has exactly $\binom{s}{n-1}$ -generators in degree $s - (n - 1)$. Thus $\mathbf{H}_{\mathbb{X}}(i) = \binom{i+n}{n}$ for $i \leq s - n$. Moreover, since

$$\mathbf{H}_{\mathbb{X}}(s - (n - 1)) = \binom{s+1}{n} - \binom{s}{n-1} = \binom{(s-n)+n}{n} = \deg(\mathbb{X}),$$

we see that \mathbb{X} has generic Hilbert function, as we wished. \square

The following example shows that the Hilbert function of a star-configuration in \mathbb{P}^n of type (n, s) is not generic in general.

Example 2.6. Consider a star-configuration in \mathbb{P}^3 of type $(3, 3)$ defined by 3 general quadratic forms. In this case, $I_{\mathbb{X}}$ has exactly 3 generators in degree 2, i.e., $\mathbf{H}_{\mathbb{X}}(2) = \binom{5}{3} - 3 = 7$. But $\deg(\mathbb{X}) = 8$, and so the Hilbert function of \mathbb{X} is

$$\mathbf{H}_{\mathbb{X}} : 1 \quad 4 \quad 7 \quad 8 \rightarrow,$$

which is not generic.

3. THE MINIMAL FREE RESOLUTION OF A STAR-CONFIGURATION IN \mathbb{P}^n

We first recall from [12] the following result.

Proposition 3.1. *Let I_C be a saturated ideal defining a codimension c subscheme $C \subseteq \mathbb{P}^n$. Let $I_S \subset I_C$ be an ideal which defines an aCM subscheme S of codimension $c - 1$. Let F be a form of degree d which is not a zero divisor on R/I_S . Consider the ideal $I' = F \cdot I_C + I_S$ and let C' be the subscheme it defines. Then I' is saturated, hence equal to $I_{C'}$, and there is an exact sequence*

$$0 \rightarrow I_S(-d) \rightarrow I_C(-d) \oplus I_S \rightarrow I_{C'} \rightarrow 0.$$

In particular, since S is an aCM subscheme of codimension one less than C , we see that C' is an aCM subscheme if and only if C is. Also

$$\deg C' = \deg C + (\deg F) \cdot (\deg S).$$

Furthermore, as sets on S , we have $C' = C' \cup H_F$, where H_F is the hyper surface section cut out on S by F . The Hilbert function $\mathbf{H}_{C'}$ of $R/I_{C'}$ is

$$\mathbf{H}_{C'}(t) = \mathbf{H}_S(t) - \mathbf{H}_S(t - d) + \mathbf{H}_C(t - d).$$

Remark 3.2. The construction in Proposition 3.1 is often referred to as *Basic Double G-Linkage*.

Theorem 3.3 ([2, Theorem 2.1]). *Let \mathbb{X} be a star-configuration in \mathbb{P}^n of type $(2, s)$ defined by general forms in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ of degrees d_1, \dots, d_s , and let $d = d_1 + d_2 + \dots + d_s$. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \rightarrow R^{s-1}(-d) \rightarrow \bigoplus_{i=1}^s R(-(d - d_i)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

Remark 3.4. Let L_1, \dots, L_s be general linear forms in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$, and let \mathbb{Y} be a star-configuration in \mathbb{P}^n of type $(2, s)$ defined by forms $L_1^{d_1}, \dots, L_s^{d_s}$ of degrees d_1, \dots, d_s . By Theorem 3.1 in [3], $R/I_{\mathbb{Y}}$ has the same minimal free resolution as $R/I_{\mathbb{X}}$, where \mathbb{X} is a star-configuration in \mathbb{P}^n of type $(2, s)$ defined by general forms F_1, \dots, F_s in R of degrees d_1, \dots, d_s , respectively. Using this result, we obtain the following corollary for a specific case.

Corollary 3.5. Let \mathbb{X} be a star-configuration in \mathbb{P}^n of type $(2, s)$ defined by general forms F_1, \dots, F_s in R of degree d , and let \mathbb{Y} be a star-configuration in \mathbb{P}^n of type $(2, s)$ defined by forms L_1^d, \dots, L_s^d , where L_i 's are general linear forms in R . Then $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ have the same minimal free resolution.

Theorem 3.6. Let $\mathbb{X}^{(r,s)}$ be a star-configuration in \mathbb{P}^n of type (r, s) defined by general forms F_1, \dots, F_s in $R = \mathbb{K}[x_0, x_1, \dots, x_n]$ of degrees d_1, d_2, \dots, d_s , where $2 \leq r \leq \min\{s, n\}$, and let $d = d_1 + \dots + d_s$. Then the minimal free resolution of $I_{\mathbb{X}^{(r,s)}}$ is

$$0 \rightarrow \mathbb{F}_r^{(r,s)} \rightarrow \mathbb{F}_{r-1}^{(r,s)} \rightarrow \dots \rightarrow \mathbb{F}_1^{(r,s)} \rightarrow I_{\mathbb{X}^{(r,s)}} \rightarrow 0 \quad (3.1)$$

where

$$\begin{aligned} \mathbb{F}_r^{(r,s)} &= R^{\alpha_r^{(r,s)}}(-d), \\ \mathbb{F}_{r-1}^{(r,s)} &= \bigoplus_{1 \leq i_1 \leq s} R^{\alpha_{r-1}^{(r,s)}}(-(d - d_{i_1})), \\ &\vdots \\ \mathbb{F}_\ell^{(r,s)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-\ell} \leq s} R^{\alpha_\ell^{(r,s)}}(-(d - (d_{i_1} + \dots + d_{i_{r-\ell}}))), \\ &\vdots \\ \mathbb{F}_2^{(r,s)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-2} \leq s} R^{\alpha_2^{(r,s)}}(-(d - (d_{i_1} + \dots + d_{i_{r-2}}))), \quad \text{and} \\ \mathbb{F}_1^{(r,s)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-1} \leq s} R(-(d - (d_{i_1} + \dots + d_{i_{r-1}}))), \end{aligned}$$

with

$$\alpha_\ell^{(r,s)} = \binom{s-r+\ell-1}{\ell-1} \quad \text{and} \quad \text{rank} \mathbb{F}_\ell^{(r,s)} = \binom{s-r+\ell-1}{\ell-1} \cdot \binom{s}{r-\ell}$$

for $1 \leq \ell \leq r$. In particular, the last free module $\mathbb{F}_r^{(r,s)}$ has only one shift d , i.e., a star-configuration $\mathbb{X}^{(r,s)}$ in \mathbb{P}^n is level. Furthermore, any star-configuration $\mathbb{X}^{(r,s)}$ in \mathbb{P}^n is aCM.

Proof. We shall prove this by double induction on r and s . If $r = 2$, then, by Theorem 3.3, the result holds. Now assume $r > 2$. Let $\mathbb{X}^{(r,s-1)}$ be a star-configuration in \mathbb{P}^n of type $(r, s-1)$ and $\mathbb{X}^{(r-1,s-1)}$ a star-configuration in \mathbb{P}^n of type $(r-1, s-1)$ defined by general forms of degrees d_1, \dots, d_{s-1} , respectively.

For the initial case of s , i.e., $r = s$, the minimal free resolution of $I_{\mathbb{X}^{(r,s)}}$ is obtained from the Koszul-complex generated by a regular sequence of general forms of degrees d_1, \dots, d_s , and so

$$\alpha_\ell^{(r,r)} = 1 = \binom{\ell-1}{\ell-1} \quad \text{and} \quad \text{rank} \mathbb{F}_\ell^{(r,r)} = \binom{r}{r-\ell} = \binom{\ell-1}{\ell-1} \cdot \binom{r}{r-\ell},$$

as we wished.

Now suppose $r < s$. By Proposition 3.1, we obtain the exact sequence

$$0 \rightarrow I_{\mathbb{X}^{(r-1,s-1)}}(-d_s) \rightarrow I_{\mathbb{X}^{(r,s-1)}}(-d_s) \oplus I_{\mathbb{X}^{(r-1,s-1)}} \rightarrow I \rightarrow 0, \quad (3.2)$$

where

$$I = F_s \cdot I_{\mathbb{X}^{(r,s-1)}} + I_{\mathbb{X}^{(r-1,s-1)}}.$$

Notice that, by Corollary 2.4, I is the ideal of a star-configuration $\mathbb{X}^{(r,s)}$ in \mathbb{P}^n of type (r, s) defined by general forms F_1, \dots, F_s in R of degrees d_1, \dots, d_s , i.e., $I = I_{\mathbb{X}^{(r,s)}}$. Let $d' = d_1 + \dots + d_{s-1}$. By double induction on r and s , we assume that

$$\begin{aligned} 0 \rightarrow \mathbb{F}_{r-1}^{(r-1,s-1)} \rightarrow \dots \rightarrow \mathbb{F}_1^{(r-1,s-1)} \rightarrow I_{\mathbb{X}^{(r-1,s-1)}} \rightarrow 0, \quad \text{and} \\ 0 \rightarrow \mathbb{F}_r^{(r,s-1)} \rightarrow \dots \rightarrow \mathbb{F}_1^{(r,s-1)} \rightarrow I_{\mathbb{X}^{(r,s-1)}} \rightarrow 0 \end{aligned}$$

are free resolutions of $\mathbb{X}^{(r-1,s-1)}$ and $\mathbb{X}^{(r,s-1)}$, respectively, such that

$$\begin{aligned}
\mathbb{F}_{r-1}^{(r-1,s-1)} &= R^{\alpha_{r-1}^{(r-1,s-1)}}(-d'), \\
\mathbb{F}_{r-2}^{(r-1,s-1)} &= \bigoplus_{1 \leq i_1 \leq s-1} R^{\alpha_{r-2}^{(r-1,s-1)}}(-(d' - d_{i_1})), \\
&\vdots \\
\mathbb{F}_{\ell}^{(r-1,s-1)} &= \bigoplus_{1 \leq i_1 < \dots < i_{(r-1)-\ell} \leq s-1} R^{\alpha_{\ell}^{(r-1,s-1)}}(-(d' - (d_{i_1} + \dots + d_{i_{(r-1)-\ell}}))), \\
&\vdots \\
\mathbb{F}_2^{(r-1,s-1)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-3} \leq s-1} R^{\alpha_2^{(r-1,s-1)}}(-(d' - (d_{i_1} + \dots + d_{i_{r-3}}))), \\
\mathbb{F}_1^{(r-1,s-1)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-2} \leq s-1} R(-(d' - (d_{i_1} + \dots + d_{i_{r-2}}))),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{F}_r^{(r,s-1)} &= R^{\alpha_r^{(r,s-1)}}(-d'), \\
\mathbb{F}_{r-1}^{(r,s-1)} &= \bigoplus_{1 \leq i_1 \leq s-1} R^{\alpha_{r-1}^{(r,s-1)}}(-(d' - d_{i_1})), \\
&\vdots \\
\mathbb{F}_{\ell}^{(r,s-1)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-\ell} \leq s-1} R^{\alpha_{\ell}^{(r,s-1)}}(-(d' - (d_{i_1} + \dots + d_{i_{r-\ell}}))), \\
&\vdots \\
\mathbb{F}_2^{(r,s-1)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-2} \leq s-1} R^{\alpha_2^{(r,s-1)}}(-(d' - (d_{i_1} + \dots + d_{i_{r-2}}))), \\
\mathbb{F}_1^{(r,s-1)} &= \bigoplus_{1 \leq i_1 < \dots < i_{r-1} \leq s-1} R(-(d' - (d_{i_1} + \dots + d_{i_{r-1}}))).
\end{aligned}$$

By the mapping cone construction with equation (3.2), we obtain the following diagram:

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \mathbb{F}_r^{(r,s-1)}(-d_s) & & & \\
& & & \downarrow & & & \\
& & & \mathbb{F}_{r-1}^{(r,s-1)}(-d_s) \oplus \mathbb{F}_{r-1}^{(r-1,s-1)} & & & \\
& & & \downarrow & & & \\
& & & \vdots & & & \\
& & & \mathbb{F}_1^{(r,s-1)}(-d_s) \oplus \mathbb{F}_1^{(r-1,s-1)} & & & \\
& & & \downarrow & & & \\
0 & \rightarrow & I_{\mathbb{X}^{(r-1,s-1)}}(-d_s) & \rightarrow & I_{\mathbb{X}^{(r,s-1)}}(-d_s) \oplus I_{\mathbb{X}^{(r-1,s-1)}} & \rightarrow & I \rightarrow 0. \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Hence we obtain a free resolution of $I = I_{\mathbb{X}}^{(r,s)}$ as

$$\begin{aligned}
0 \rightarrow & \begin{bmatrix} \mathbb{F}_r^{(r,s-1)}(-d_s) \\ \oplus \\ \mathbb{F}_{r-1}^{(r-1,s-1)}(-d_s) \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{F}_{r-1}^{(r,s-1)}(-d_s) \oplus \mathbb{F}_{r-1}^{(r-1,s-1)} \\ \oplus \\ \mathbb{F}_{r-2}^{(r-1,s-1)}(-d_s) \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} \mathbb{F}_2^{(r,s-1)}(-d_s) \oplus \mathbb{F}_2^{(r-1,s-1)} \\ \oplus \\ \mathbb{F}_1^{(r-1,s-1)}(-d_s) \end{bmatrix} \\
& \rightarrow \mathbb{F}_1^{(r,s-1)}(-d_s) \oplus \mathbb{F}_1^{(r-1,s-1)} \rightarrow I \rightarrow 0.
\end{aligned} \tag{3.3}$$

Now consider a free module

$$\begin{aligned}
\mathbb{F}_\ell^{(r,s)} &= \mathbb{F}_\ell^{(r,s-1)}(-d_s) \oplus \mathbb{F}_\ell^{(r-1,s-1)} \oplus \mathbb{F}_{\ell-1}^{(r-1,s-1)}(-d_s), \\
&= \left[\begin{aligned} & \bigoplus_{1 \leq i_1 < \cdots < i_{r-\ell} \leq s-1} R^{\alpha_\ell^{(r,s-1)}}(-(d' + d_s - (d_{i_1} + \cdots + d_{i_{r-\ell}}))) \\ & \oplus \\ & \bigoplus_{1 \leq i_1 < \cdots < i_{(r-1)-\ell} \leq s-1} R^{\alpha_\ell^{(r-1,s-1)}}(-(d' - (d_{i_1} + \cdots + d_{i_{(r-1)-\ell}}))) \\ & \oplus \\ & \bigoplus_{1 \leq i_1 < \cdots < i_{(r-1)-(\ell-1)} \leq s-1} R^{\alpha_{\ell-1}^{(r-1,s-1)}}(-(d' + d_s - (d_{i_1} + \cdots + d_{i_{(r-1)-(\ell-1)}}))). \end{aligned} \right]
\end{aligned} \tag{3.4}$$

for $1 \leq \ell \leq s$. Since $d = d' + d_s$, we can rewrite equation (3.4) as

$$\begin{aligned}
\mathbb{F}_\ell^{(r,s)} &= \left[\begin{aligned} & \bigoplus_{1 \leq i_1 < \cdots < i_{r-\ell} \leq s-1} R^{\alpha_\ell^{(r,s-1)}}(-(d - (d_{i_1} + \cdots + d_{i_{r-\ell}}))) \\ & \oplus \\ & \bigoplus_{1 \leq i_1 < \cdots < i_{(r-1)-\ell} \leq s-1} R^{\alpha_\ell^{(r-1,s-1)}}(-(d - (d_{i_1} + \cdots + d_{i_{(r-1)-\ell}}) + d_s))) \\ & \oplus \\ & \bigoplus_{1 \leq i_1 < \cdots < i_{r-\ell} \leq s-1} R^{\alpha_{\ell-1}^{(r-1,s-1)}}(-(d - (d_{i_1} + \cdots + d_{i_{r-\ell}}))) \\ & \oplus \\ & \bigoplus_{1 \leq i_1 < \cdots < i_{r-\ell} \leq s-1} R^{\alpha_{\ell-1}^{(r-1,s-1)} + \alpha_\ell^{(r,s-1)}}(-(d - (d_{i_1} + \cdots + d_{i_{r-\ell}}))) \\ & \oplus \\ & \bigoplus_{1 \leq i_1 < \cdots < i_{(r-1)-\ell} \leq s-1} R^{\alpha_\ell^{(r-1,s-1)}}(-(d - (d_{i_1} + \cdots + d_{i_{(r-1)-\ell}}) + d_s))) \end{aligned} \right].
\end{aligned} \tag{3.5}$$

Now we shall prove that $\alpha_\ell^{(r,s)} := \alpha_{\ell-1}^{(r-1,s-1)} + \alpha_\ell^{(r,s-1)} = \alpha_\ell^{(r-1,s-1)}$, i.e., $\mathbb{F}_\ell^{(r,s)}$ is of the form

$$\mathbb{F}_\ell^{(r,s)} = \bigoplus_{1 \leq i_1 < \cdots < i_{r-\ell} \leq s} R^{\alpha_\ell^{(r,s)}}(-(d - (d_{i_1} + \cdots + d_{i_{r-\ell}}))).$$

If $\ell = 1$, then by Corollary 2.4

$$\alpha_1^{(r,s)} = 1 = \binom{s-r+1-1}{1-1}, \quad \text{i.e.,} \quad \text{rank} \mathbb{F}_1 = \binom{s}{r-1} = \binom{s}{r-1} \cdot \binom{s-r+1-1}{1-1}.$$

Now suppose $1 < \ell < r$. Then by double induction on r and s , we have that

$$\begin{aligned}
\alpha_{\ell-1}^{(r-1,s-1)} + \alpha_{\ell}^{(r,s-1)} &= \binom{(s-1) - (r-1) + (\ell-1) - 1}{(\ell-1) - 1} + \binom{(s-1) - r + \ell - 1}{\ell-1} \\
&= \binom{s-r+\ell-2}{\ell-2} + \binom{s-r+\ell-2}{\ell-1} \\
&= \binom{s-r+\ell-1}{\ell-1} \\
&= \binom{(s-1) - (r-1) + \ell - 1}{\ell-1} \\
&= \alpha_{\ell}^{(r-1,s-1)},
\end{aligned}$$

and hence

$$\text{rank } \mathbb{F}_{\ell} = \binom{s-r+\ell-1}{\ell-1} \cdot \binom{s}{r-\ell}.$$

Moreover, if $\ell = r$, then it is from equation (3.3) and double induction on r and s that

$$\begin{aligned}
\alpha_r^{(r,s)} &= \alpha_r^{(r,s-1)} + \alpha_{r-1}^{(r-1,s-1)} \\
&= \binom{s-2}{r-1} + \binom{s-2}{r-2} \\
&= \binom{s-1}{r-1} \\
&= \text{rank } \mathbb{F}_r^{(r,s)},
\end{aligned}$$

as we wished.

Now we shall prove that the free resolution in equation (3.3) is minimal. Since \mathbb{k} is an infinite field, for any $1 \leq r \leq \min\{s, n\}$, we can take an $(s-r+1) \times s$ matrix $A = [a_{i,j}]$ such that any $\gamma \times \gamma$ minor is not 0 for some $a_{i,j} \in \mathbb{k}$ and for every $1 \leq \gamma \leq s-r+1$. Define a $(s-r+1) \times s$ matrix $M = [a_{i,j}F_j]$. Then, a $\gamma \times \gamma$ -minor of M is of the form

$$\begin{aligned}
&\det \begin{bmatrix} a_{i_1 1, j_1} F_{j_1} & \cdots & a_{i_1, j_{\gamma}} F_{j_{\gamma}} \\ & \cdots & \\ a_{i_{\gamma}, j_1} F_{j_1} & \cdots & a_{i_{\gamma}, j_{\gamma}} F_{j_{\gamma}} \end{bmatrix} \\
&= \det \begin{bmatrix} a_{i_1 1, j_1} & \cdots & a_{i_1, j_{\gamma}} \\ & \cdots & \\ a_{i_{\gamma}, j_1} & \cdots & a_{i_{\gamma}, j_{\gamma}} \end{bmatrix} \cdot F_{j_1} \cdots F_{j_{\gamma}}.
\end{aligned}$$

Since any $\gamma \times \gamma$ minor of the matrix A is not 0, by Corollary 2.4 we get that the ideal generated by all maximal minors of the matrix M is $I_{\mathbb{X}}$. Since $\text{depth } I_{\mathbb{X}} = r$, by Corollary A2.12 in [8] the graded Betti numbers of the homogeneous coordinate ring of $\mathbb{X} := \mathbb{X}^{(r,s)}$ are those given by the Eagon-Northcott resolution of the $\gamma \times \gamma$ minors of M . In other words, the free resolution of $I_{\mathbb{X}}$ in equation (3.3) is minimal. Moreover, since the last free module $\mathbb{F}_r^{(r,s)}$ has only one shift d , any star-configuration $\mathbb{X}^{(r,s)}$ in \mathbb{P}^n is level.

For the last assertion, recall that, by Theorem 3.3, any star-configuration in \mathbb{P}^n of type $(2, s)$ (i.e., codimension 2) is aCM. Suppose $r > 2$. If $r = s$, then $\mathbb{X}^{(r,s)}$ is a complete intersection, i.e., $\mathbb{X}^{(r,s)}$ is aCM. If $r < s$, then by double induction on r and s , we assume that $\mathbb{X}^{(r-1,s-1)}$ and $\mathbb{X}^{(r,s-1)}$ are aCM, and so, by Proposition 3.1, $\mathbb{X}^{(r,s)}$ is also aCM, which completes this theorem. \square

As a corollary, we obtain the result in [9] with $d_1 = \cdots = d_s = 1$ in Theorem 3.6 and we omit the proof.

Corollary 3.7 ([9, Remark 2.11]). *Let $\mathbb{X} := \mathbb{X}^{(r,s)}$ be a linear star-configuration in \mathbb{P}^n of type (r, s) with $1 \leq r \leq \min\{s, n\}$. Then the minimal free resolution of $I_{\mathbb{X}}$ is given by*

$$0 \rightarrow \mathbb{F}_r^{(r,s)} \rightarrow \mathbb{F}_{r-1}^{(r,s)} \rightarrow \cdots \rightarrow \mathbb{F}_1^{(r,s)} \rightarrow I_{\mathbb{X}^{(r,s)}} \rightarrow 0$$

in Theorem 3.6 with $d_1 = \cdots = d_s = 1$.

4. THE WEAK-LEFSCHETZ PROPERTY

A graded Artinian \mathbb{k} -algebra $A = \bigoplus_{i=0}^s A_i$ ($A_s \neq 0$) has the *weak-Lefschetz property* if the homomorphism $(\times L) : A_i \rightarrow A_{i+1}$, induced by multiplication by a linear form L , has maximal rank for each i . In this case, we call L a *Lefschetz element*.

We first recall a question in [2].

Question 4.1 ([2, Question 1.3]). Let $\mathbb{X} := \mathbb{X}^{(2,s)}$ and $\mathbb{Y} := \mathbb{X}^{(2,t)}$ be star-configurations in \mathbb{P}^2 of type $(2, s)$ and $(2, t)$ defined by general forms of degree $d \geq 1$ with $s \geq 3$. Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak-Lefschetz property?

We revise the above question to a more general question as follows.

Question 4.2. Let $\mathbb{X} := \mathbb{X}^{(n,s)}$ and $\mathbb{Y} := \mathbb{X}^{(n,t)}$ be star-configurations in \mathbb{P}^n of type (n, s) and (n, t) defined by general forms of degree $d \geq 1$. Does the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak-Lefschetz property?

We start with a proposition on the weak-Lefschetz property from [10] and provide an answer to Question 4.2 for $d = 1$. Let \mathbb{X} be a finite set of points in \mathbb{P}^n and define

$$\sigma(\mathbb{X}) = \min\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

Proposition 4.3 ([10, Proposition 5.15]). *Let \mathbb{X} be a finite set of points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring of \mathbb{X} . Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for all $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Then A has the weak-Lefschetz property.*

By the same argument as in the proof of Theorem 4.2 in [2], we obtain the following theorem immediately and thus omit the proof.

Theorem 4.4. *Let $\mathbb{X} := \mathbb{X}^{(n,s)}$ and $\mathbb{Y} := \mathbb{X}^{(n,t)}$ be linear star configurations in \mathbb{P}^n of type (n, t) and (n, s) with $s \geq t \geq n$, respectively. Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property.*

We also recall the following remark in [2].

Remark 4.5 ([2, Remark 4.3]). If $\mathbb{X}^{(n,s)}$ and $\mathbb{X}^{(n,t)}$ are not linear star-configurations in Theorem 4.4, then Theorem 4.4 may not hold in general. For example, assume that $\mathbb{X} := \mathbb{X}^{(2,4)}$ and $\mathbb{Y} := \mathbb{X}^{(2,4)}$ are star-configurations in \mathbb{P}^2 of type $(2, 4)$ defined by general forms in $R = \mathbb{k}[x_0, x_1, x_2]$ of degree 2. Then, by Theorem 3.6, the Hilbert functions of $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ are

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 24 \rightarrow,$$

and thus

$$\sigma(\mathbb{X}) = \sigma(\mathbb{Y}) = 7.$$

Furthermore, the Hilbert function of $R/I_{\mathbb{X} \cup \mathbb{Y}}$, obtained by CoCoA, is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad 45 \quad 48 \rightarrow,$$

and thus

$$\begin{aligned} \mathbf{H}(R/I_{\mathbb{X}} + I_{\mathbb{Y}}, 6) &= \mathbf{H}(R/I_{\mathbb{X}}, 6) + \mathbf{H}(R/I_{\mathbb{Y}}, 6) - \mathbf{H}(R/I_{\mathbb{X} \cup \mathbb{Y}}, 6) \\ &= 24 + 24 - 28 \\ &= 20 \\ &\neq \mathbf{H}(R/I_{\mathbb{X}}, 6). \end{aligned}$$

This does not satisfy the conditions in Proposition 4.3, and thus we do not know if Theorem 4.4 still holds for this case when \mathbb{X} and \mathbb{Y} are star-configurations in \mathbb{P}^n defined by general forms of degree d with $d \geq 2$.

Theorem 4.4 gives a complete answer to Question 4.2 for $d = 1$. In other words, Question 4.1 for $d > 1$ is still open. Thus, we restate Question 4.1 as follows.

Question 4.6 (Restated Question 4.2). Let $\mathbb{X} := \mathbb{X}^{(n,s)}$ and $\mathbb{Y} := \mathbb{X}^{(n,t)}$ be star-configurations in \mathbb{P}^n of type (n, s) and (n, t) , respectively, defined by general forms of degree $d > 1$. Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak-Lefschetz property?

Furthermore, we have the following question in general.

Question 4.7. Let $\mathbb{X} := \mathbb{X}^{(n,s)}$ and $\mathbb{Y} := \mathbb{X}^{(n,t)}$ be star-configurations in \mathbb{P}^n of type (n, s) and (n, t) defined by general forms of degrees d_1, \dots, d_s and d'_1, \dots, d'_t , respectively. Does an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak-Lefschetz property?

Now we move on to a more general case of the union of two star-configurations in \mathbb{P}^2 of type $(2, s)$ with $s \geq 2$. In [18], the author found that if \mathbb{X} and \mathbb{Y} are star-configurations in \mathbb{P}^2 defined by general forms of degrees ≤ 2 and $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property.

Theorem 4.8 ([18, Theorem 3.3]). *Let $\mathbb{X} := \mathbb{X}^{(2,s)}$ and $\mathbb{Y} := \mathbb{X}^{(2,t)}$ be star-configurations in \mathbb{P}^2 of type $(2, s)$ and $(2, t)$ defined by general forms F_1, \dots, F_s and G_1, \dots, G_t in $R = \mathbb{k}[x_0, x_1, x_2]$, respectively, with $s, t \geq 3$. Assume $\deg(F_i) \leq 2$ for $i = 1, \dots, s$ and $\deg(G_j) \leq 2$ for $j = 1, \dots, t$. If $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property.*

Before we introduce a new Artinian quotient of a coordinate ring of a star-configuration in \mathbb{P}^2 having the weak-Lefschetz property without the condition $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, we need the following two propositions.

Proposition 4.9 ([18, Proposition 3.6]). *Let $\mathbb{X}^{(2,s)}$ be a star-configuration in \mathbb{P}^2 defined by general forms F_1, \dots, F_s of degrees $1 \leq d_1 \leq \dots \leq d_s$ with $s \geq 3$. Then*

$$\sigma(\mathbb{X}^{(2,s)}) = \left[\sum_{i=1}^s d_i \right] - 1.$$

Proposition 4.10 ([18, Proposition 2.6]). *If $\mathbb{X}^{(2,s)}$ is a star-configuration in \mathbb{P}^2 of type $(2, s)$ defined by general forms F_1, \dots, F_s of degrees $1 \leq d_1 \leq \dots \leq d_s \leq 2$ with $s \geq 3$, then $\mathbb{X}^{(2,s)}$ has generic Hilbert function. In particular, the Hilbert function of $\mathbb{X}^{(2,s)}$ is*

$$\mathbf{H}_{\mathbb{X}} : 1 \quad \binom{1+2}{2} \quad \dots \quad \binom{2+(d-3)}{2} \quad \deg(\mathbb{X}^{(2,s)}) \rightarrow,$$

where $d = \sum_{j=1}^s d_j$.

We now discuss an Artinian quotient of a coordinate ring of a star-configuration in \mathbb{P}^2 having the Weak Lefschetz property without the condition $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$.

Proposition 4.11. *Let $\mathbb{X} := \mathbb{X}^{(2,s)}$ and $\mathbb{Y} := \mathbb{X}^{(2,t)}$ be as in Theorem 4.8. Assume $\deg(F_i) = 1$ for every $1 \leq i \leq s$ and $\deg(G_j) \leq 2$ for every $1 \leq j \leq t$. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property.*

Proof. If $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then it is immediate from Theorem 4.4. Now assume that $\sigma(\mathbb{X}) = \sigma(\mathbb{Y})$, i.e., $\sum_{i=1}^s \deg(F_i) = \sum_{i=1}^t \deg(G_i)$. It is from Proposition 4.10 that the Hilbert functions of \mathbb{X} and \mathbb{Y} are

$$\begin{aligned} \mathbf{H}_{\mathbb{X}} &: 1 \quad \binom{1+2}{2} \quad \dots \quad \binom{(s-3)+2}{2} \quad \deg(\mathbb{X}) \rightarrow, \quad \text{and} \\ \mathbf{H}_{\mathbb{Y}} &: 1 \quad \binom{1+2}{2} \quad \dots \quad \binom{(s-3)+2}{2} \quad \deg(\mathbb{Y}) \rightarrow. \end{aligned}$$

Note that

$$\mathbf{H}_{\mathbb{X}}(s-2) = \deg(\mathbb{X}) = \binom{2+(s-2)}{2}.$$

In other words, $I_{\mathbb{X}}$ has no generators in degree $s - 2$, and thus

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), s - 2) = \mathbf{H}_{\mathbb{Y}}(s - 2).$$

Moreover, since

$$\mathbf{H}_{\mathbb{X}}(i) = \mathbf{H}_{\mathbb{Y}}(i) = \binom{2+i}{2}$$

for $0 \leq i \leq s - 3$, and so

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), i) = \mathbf{H}_{\mathbb{Y}}(i) = \binom{2+i}{2}.$$

Hence we get that

$$\mathbf{H}(R/(I_{\mathbb{X}} + I_{\mathbb{Y}}), i) = \mathbf{H}(R/I_{\mathbb{Y}}, i)$$

for every $0 \leq i \leq s - 2 = \sigma(\mathbb{Y}) - 1$. Furthermore, since

$$R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \simeq (R/I_{\mathbb{Y}})/((I_{\mathbb{X}} + I_{\mathbb{Y}})/I_{\mathbb{Y}})$$

is an Artinian quotient of the coordinate ring $R/I_{\mathbb{Y}}$, by Proposition 4.3 $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property, as we wished. \square

By Corollary 3.5, we often use a product of linear forms L_1, \dots, L_d in $R = \mathbb{k}[x_0, \dots, x_n]$ instead of general forms F of degree d to construct a star-configuration in \mathbb{P}^n for this section. In [18], the author showed that if $\mathbb{X} := \mathbb{X}^{(2,s)}$ is a star-configuration in \mathbb{P}^2 of type $(2, s)$ defined by general quadratic forms F_1, \dots, F_s and $\mathbb{Y} := \mathbb{X}^{(2,s+1)}$ is a star-configuration in \mathbb{P}^2 of type $(2, s + 1)$ defined by general quadratic forms G_1, \dots, G_s and a general linear form L , then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property with a Lefschetz element L (see [18, Theorem 3.7]). We shall generalize this result with the condition $\deg(F_i) = \deg(G_i) \leq 2$ for every $i = 1, \dots, s$.

For the rest of this section, to distinguish two star-configurations, we shall use the following notations and symbols for lines and points in pictorial description.

$$\begin{array}{ll} \text{a solid line} & \text{---} \quad \mathbb{L}_i \quad \text{is a line defined by a linear form } L_i, \text{ and} \\ \text{a dashed line} & \text{---} \text{---} \quad \mathbb{M}_i \quad \text{is a line defined by a linear form } M_i \end{array}$$

for $1 \leq i \leq s$ with $s \geq 2$. We also define that

$$\begin{array}{ll} P_{i,j} & \text{is a point defined by linear forms } L_i, L_j, \text{ and} \\ Q_{i,j} & \text{is a point defined by linear forms } M_i, M_j \end{array}$$

where L_i, L_j and M_i, M_j are linear forms in R for $1 \leq i < j \leq s$ with $s \geq 2$.

Lemma 4.12. *Let \mathbb{X} be the union of two star-configurations $\mathbb{X}_1 := \mathbb{X}^{(2,3)}$ and $\mathbb{X}_2 := \mathbb{X}^{(2,2)}$ in \mathbb{P}^2 of type $(2, 3)$ and $(2, 2)$, respectively. Then \mathbb{X} has generic Hilbert function*

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 16 \quad \rightarrow .$$

Proof. Without loss of generality, we assume that \mathbb{X}_1 is defined by quadratic forms L_1L_2, L_3L_4 , and L_5L_6 , where L_i is a linear form defining a line \mathbb{L}_i for $i = 1, \dots, 6$, and that \mathbb{X}_2 is defined by quadratic forms M_1M_2 and M_3M_4 , where M_i is a linear form defining a line \mathbb{M}_i for $i = 1, \dots, 4$. Furthermore, we assume that L_1 vanishes on four points in \mathbb{X}_1 and one point defined by two linear forms M_1 and M_4 in \mathbb{X}_2 , and that M_2 vanishes on two points in \mathbb{X}_2 and one point defined by linear forms L_3 and L_6 in \mathbb{X}_1 (see Figure 1 again).

Let $N \in (I_{\mathbb{X}})_4$, then by *Bezout's Theorem*,

$$N = \alpha L_1L_2M_2L_5$$

for some $\alpha \in \mathbb{k}$. Moreover, since N has to vanish on two more points $P_{4,6}, Q_{1,3}$ in \mathbb{X} , where none of L_1, L_2, M_2 , and L_5 can vanish, we get that $N = 0$. Therefore, the Hilbert function of \mathbb{X} is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 16 \quad \rightarrow ,$$

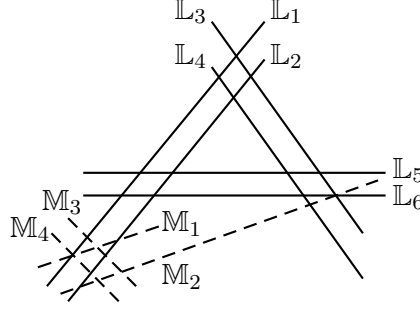


FIGURE 1.

as we wished. \square

Theorem 4.13. *Let \mathbb{X} be the union of two star-configurations $\mathbb{X}_1 := \mathbb{X}^{(2,3)}$ and $\mathbb{X}_2 := \mathbb{X}^{(2,3)}$ in \mathbb{P}^2 of type $(2, 3)$. Then \mathbb{X} has generic Hilbert function*

$$1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 24 \rightarrow .$$

Proof. Without loss of generality, we assume that \mathbb{X}_1 and \mathbb{X}_2 are defined by quadratic forms $L_i L_{i+1}$ and $M_i M_{i+1}$, respectively, for $i = 1, 3, 5$. (see Figure 2). We also assume that

- a linear form L_1 vanishes on 6 points $P_{1,3}, P_{1,4}, P_{1,5}, P_{1,6}, Q_{1,6}, Q_{2,5}$, and
- a linear form L_2 vanishes on 5 points $P_{2,3}, P_{2,4}, P_{2,5}, P_{2,6}, Q_{2,6}$.

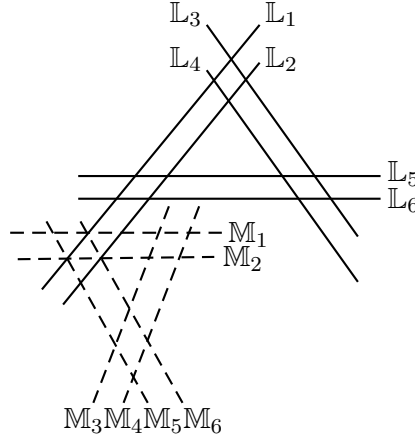


FIGURE 2.

For every $N \in R_5$, by *Bez out's theorem*,

$$N = \alpha L_1 L_2 M_3 M_4 L_5,$$

for some $\alpha \in \mathbb{k}$. Moreover, since N has to vanish on a point $P_{3,6}$, where none of L_1, L_2, M_3, M_4 , and L_5 vanishes, we have $N = 0$. Hence the Hilbert function of \mathbb{X} is of the form

$$\mathbf{H}_{\mathbb{X}} : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ \cdots . \quad (4.1)$$

Let $\mathbb{Y}_1 := \mathbb{X}_1$, and $\mathbb{Y}_2 := \mathbb{X}^{(2,2)}$ be a star-configuration in \mathbb{P}^2 of type $(2, 2)$ defined by quadratic forms $M_1 M_2$ and $M_3 M_4$, respectively. Define $\mathbb{Y} := \mathbb{Y}_1 \cup \mathbb{Y}_2$. By Lemma 4.12 the Hilbert function of \mathbb{Y} is

$$\mathbf{H}_{\mathbb{Y}} : 1 \ 3 \ 6 \ 10 \ 15 \ 16 \rightarrow . \quad (4.2)$$

Let $\mathbb{Z}_1 := \mathbb{X}_1$ and \mathbb{Z}_2 be a star-configuration in \mathbb{P}^2 of type $(2, 3)$ defined by $M_1 M_2, M_3 M_4$, and M_5 . Define $\mathbb{Z} := \mathbb{Z}_1 \cup \mathbb{Z}_2$ (see Figure 2), and let $G_4 = M_1 \cdots M_4$.

Using equation (4.2) and the exact sequence

$$0 \rightarrow R/I_{\mathbb{Z}} \rightarrow R/I_{\mathbb{Y}} \oplus R/(M_5, G) \rightarrow R/(I_{\mathbb{Y}}, M_5, G) \rightarrow 0,$$

we get that

$$\begin{aligned} \mathbf{H}(R/I_{\mathbb{Z}}, -) &: 1 & 3 & 6 & 10 & 15 & - & 20 & \rightarrow, \\ \mathbf{H}(R/I_{\mathbb{Y}}, -) &: 1 & 3 & 6 & 10 & 15 & 16 & 16 & \rightarrow, \\ \mathbf{H}(R/(M_5, G_4), -) &: 1 & 2 & 3 & 4 & 4 & 4 & 4 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{Y}}, M_5, G_4), -) &: 1 & 2 & 3 & 4 & 4 & - & 0 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{Y}}, M_5), -) &: 1 & 2 & 3 & 4 & 5 & 1 & 0 & \rightarrow. \end{aligned}$$

Moreover, since $\mathbf{H}(R/(I_{\mathbb{Y}}, M_5), 5) = 1$, we see that

$$\mathbf{H}(R/I_{\mathbb{Y}}, M_5, G_4), 5) = 0, \text{ or } 1.$$

In other words, there are only two possible Hilbert functions for $R/I_{\mathbb{Z}}$:

$$\begin{aligned} (1) &: 1 & 3 & 6 & 10 & 15 & 20 & 20 & \rightarrow, & \text{ or} \\ (2) &: 1 & 3 & 6 & 10 & 15 & 19 & 20 & \rightarrow. \end{aligned}$$

However, using the same argument as in the proof of Theorem 3.3 in [17], one can show that (1) is the Hilbert function of \mathbb{Z} .

Recall that \mathbb{X}_1 and \mathbb{X}_2 are star-configurations in \mathbb{P}^2 of type $(2, 3)$ defined by quadratic forms $L_i L_{i+1}$ and $M_i M_{i+1}$, respectively, for $i = 1, 3, 5$. Let $\mathbb{X} := \mathbb{X}_1 \cup \mathbb{X}_2$ and $G_4 = M_1 \cdots M_4$.

By equation (4.1) and the following exact sequence

$$0 \rightarrow R/I_{\mathbb{X}} \rightarrow R/I_{\mathbb{Z}} \oplus R/(M_6, G) \rightarrow R/(I_{\mathbb{Z}}, M_6, G) \rightarrow 0,$$

we get that

$$\begin{aligned} \mathbf{H}(R/I_{\mathbb{X}}, -) &: 1 & 3 & 6 & 10 & 15 & 21 & - & 24 & \rightarrow, \\ \mathbf{H}(R/I_{\mathbb{Z}}, -) &: 1 & 3 & 6 & 10 & 15 & 20 & 20 & 20 & \rightarrow, \\ \mathbf{H}(R/(M_6, G_4), -) &: 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{Z}}, M_6, G_4), -) &: 1 & 2 & 3 & 4 & 4 & 3 & - & 0 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{Z}}, M_6), -) &: 1 & 2 & 3 & 4 & 5 & 5 & 0 & 0 & \rightarrow. \end{aligned}$$

Moreover, since $\mathbf{H}(R/(I_{\mathbb{Z}}, M_6), 6) = 0$, we see that $\mathbf{H}(R/I_{\mathbb{Z}}, M_6, G_4), 6) = 0$ as well. Thus the Hilbert function of \mathbb{X} is

$$\mathbf{H}(R/I_{\mathbb{X}}, -) : 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 24 \rightarrow,$$

as desired. \square

By *Bezout's* Theorem and Theorem 4.13, we obtain the following proposition by double induction on d and s and thus omit the proof.

Proposition 4.14. *Let \mathbb{X} be the union of two star-configurations in \mathbb{P}^2 of type $(2, s)$ defined by general forms in $R = \mathbb{k}[x_0, x_1, x_2]$ of degree d with $s \geq 4$ and $d \geq 2$. Then*

$$(I_{\mathbb{X}})_{ds} = \{0\}.$$

Lemma 4.15. *Let \mathbb{X} and \mathbb{Y} be star-configurations in \mathbb{P}^2 of type $(2, s)$ defined by general forms F_1, \dots, F_s and G_1, \dots, G_s with $s \geq 3$, respectively. Assume that*

$$\deg(F_i) = \deg(G_i) = \begin{cases} 1, & \text{for } i = 1, \dots, \ell, \\ 2, & \text{for } i = \ell + 1, \dots, s, \end{cases}$$

with $0 \leq \ell < s$. Then

$$\dim_{\mathbb{k}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-2} = 2(s - \ell).$$

Proof. We shall prove this by induction on s . If $s \geq 3$ and $\ell = 0$, then by Propositions 4.13 and 4.14, and the exact sequence

$$0 \rightarrow R/I_{\mathbb{X} \cup \mathbb{Y}} \rightarrow R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \rightarrow R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \rightarrow 0, \quad (4.3)$$

we have

$$\dim_{\mathbb{K}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-2} = \dim_{\mathbb{K}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-2} = 2s = 2(s - \ell).$$

Assume $1 \leq \ell < s$. If $s = 3$, then using the same idea as in the proof of Theorem 4.13, one can prove that

$$\dim_{\mathbb{K}}(I_{\mathbb{X} \cup \mathbb{Y}})_{4-\ell} = \dim_{\mathbb{K}}(I_{\mathbb{X} \cup \mathbb{Y}})_{2s-\ell-2} = 0,$$

and thus

$$\dim_{\mathbb{K}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-2} = \dim_{\mathbb{K}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{4-\ell} = 2(3 - \ell) = 2(s - \ell).$$

Now suppose $s > 3$. Let \mathbb{F}_1 be a line defined by a linear form F_1 and $N \in (I_{\mathbb{X} \cup \mathbb{Y}})_{2s-\ell-2}$. Notice that F_1 vanishes on $(2s - \ell - 1)$ -points on a line \mathbb{F}_1 . By *Bezout's* Theorem, we have

$$N = F_1 N'$$

for some $N' \in R_{2s-\ell-3}$. Let \mathbb{X}' and \mathbb{Y}' be star-configurations in \mathbb{P}^2 of type $(2, s - 1)$ defined by F_2, \dots, F_s and G_2, \dots, G_s , respectively. Then,

$$\begin{aligned} N' &\in (I_{\mathbb{X}' \cup \mathbb{Y}'})_{2s-\ell-3} \\ &= (I_{\mathbb{X}' \cup \mathbb{Y}'})_{2(s-1)-(\ell-1)-2} \\ &= \{0\}, \quad (\text{by induction on } s) \end{aligned}$$

and thus $(I_{\mathbb{X} \cup \mathbb{Y}})_{2s-\ell-2} = \{0\}$ as well. Therefore, using equation (4.3), we get that

$$\dim_{\mathbb{K}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-2} = 2(s - \ell),$$

which completes the proof of this lemma. \square

Remark 4.16. Let \mathbb{X} and \mathbb{Y} be as in Lemma 4.15. Since $(I_{\mathbb{X} \cup \mathbb{Y}})_{2s-\ell-2} = \langle \tilde{F}_{\ell+1}, \dots, \tilde{F}_s, \tilde{G}_{\ell+1}, \dots, \tilde{G}_s \rangle$ and by Lemma 4.15, $\dim_{\mathbb{K}}(I_{\mathbb{X} \cup \mathbb{Y}})_{2s-\ell-2} = 2(s - \ell)$, we see that the following $2(s - \ell)$ -forms

$$\tilde{F}_{\ell+1}, \dots, \tilde{F}_s, \tilde{G}_{\ell+1}, \dots, \tilde{G}_s$$

are linearly independent.

We are now ready to prove the following theorem, which generalises Theorem 3.7 in [18].

Theorem 4.17. Let \mathbb{X} be a star-configuration in \mathbb{P}^2 of type $(2, s)$ defined by general forms F_1, \dots, F_s and \mathbb{Y} be a star-configuration in \mathbb{P}^2 of type $(2, s + 1)$ defined by general forms G_1, \dots, G_s and a general linear form L . Assume that

$$\deg(F_i) = \deg(G_i) = \begin{cases} 1, & \text{for } i = 1, \dots, \ell, \\ 2, & \text{for } i = \ell + 1, \dots, s, \end{cases}$$

where $0 \leq \ell < s$. Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property with a Lefschetz element L .

Proof. First, by Proposition 4.9, we have $\sigma(\mathbb{X}) < \sigma(\mathbb{Y})$, and hence by Theorem 4.8, $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak-Lefschetz property. It suffices to show that L is a Lefschetz element.

Note that the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is of the form:

$$\mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})}(-) : 1 \quad \binom{1+2}{2} \quad \dots \quad \binom{(2s-\ell-3)+2}{2} \quad \binom{(2s-\ell-2)+2}{2} - (s-\ell) \quad \dots. \quad (4.4)$$

We shall show that

$$\dim_{\mathbb{K}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-1} = 4s - 3\ell, \quad \text{i.e.,} \quad \mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})}(2s - \ell - 1) = \binom{(2s-\ell-1)+2}{2} - (4s - 3\ell).$$

Consider the following $(4s - 3\ell)$ -forms in $(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-1}$

$$\tilde{F}_1, \dots, \tilde{F}_\ell, x_0 \tilde{F}_{\ell+1}, x_1 \tilde{F}_{\ell+1}, x_2 \tilde{F}_{\ell+1}, \dots, x_0 \tilde{F}_s, x_1 \tilde{F}_s, x_2 \tilde{F}_s, L \tilde{G}_{\ell+1}, \dots, L \tilde{G}_s,$$

where

$$\tilde{F}_i = \frac{\prod_{j=1}^s F_j}{F_i} \text{ and } \tilde{G}_i = \frac{\prod_{j=1}^s G_j}{G_i},$$

for $i = 1, \dots, s$.

Suppose that

$$\alpha_1 \tilde{F}_1 + \dots + \alpha_\ell \tilde{F}_\ell + \alpha_{0\ell+1} x_0 \tilde{F}_{\ell+1} + \alpha_{1\ell+1} x_1 \tilde{F}_{\ell+1} + \alpha_{2\ell+1} x_2 \tilde{F}_{\ell+1} + \dots + \alpha_{0s} x_0 \tilde{F}_s + \alpha_{1s} x_1 \tilde{F}_s + \alpha_{2s} x_2 \tilde{F}_s + \beta_{\ell+1} L \tilde{G}_{\ell+1} + \dots + \beta_s L \tilde{G}_s = 0, \quad (4.5)$$

where $\alpha_i, \alpha_{ij}, \beta_i \in \mathbb{k}$ for every i, j . Since L is a linear factor of the form

$$\alpha_1 \tilde{F}_1 + \dots + \alpha_\ell \tilde{F}_\ell + \alpha_{0\ell+1} x_0 \tilde{F}_{\ell+1} + \alpha_{1\ell+1} x_1 \tilde{F}_{\ell+1} + \alpha_{2\ell+1} x_2 \tilde{F}_{\ell+1} + \dots + \alpha_{0s} x_0 \tilde{F}_s + \alpha_{1s} x_1 \tilde{F}_s + \alpha_{2s} x_2 \tilde{F}_s,$$

and L is a non-zero divisor of $R/I_{\mathbb{X}}$, we see that

$$\beta_{\ell+1} \tilde{G}_{\ell+1} + \dots + \beta_s \tilde{G}_s \in (I_{\mathbb{X}})_{2s-\ell-2}.$$

Moreover, by Lemma 4.15 (see also Remark 4.16)

$$\beta_{\ell+1} = \dots = \beta_s = 0.$$

Thus we rewrite equation (4.5) as

$$\alpha_1 \tilde{F}_1 + \dots + \alpha_\ell \tilde{F}_\ell + \alpha_{0\ell+1} x_0 \tilde{F}_{\ell+1} + \alpha_{1\ell+1} x_1 \tilde{F}_{\ell+1} + \alpha_{2\ell+1} x_2 \tilde{F}_{\ell+1} + \dots + \alpha_{0s} x_0 \tilde{F}_s + \alpha_{1s} x_1 \tilde{F}_s + \alpha_{2s} x_2 \tilde{F}_s = 0. \quad (4.6)$$

Note that $F_j \mid \tilde{F}_i$ for every $j \neq i$. Hence

$$F_1 \mid \alpha_1 \tilde{F}_1,$$

and so $\alpha_1 = 0$. By the same method as above, one can show that $\alpha_1 = \dots = \alpha_\ell = 0$. Moreover, since

$$F_{\ell+1} \mid (\alpha_{0\ell+1} x_0 \tilde{F}_{\ell+1} + \alpha_{1\ell+1} x_1 \tilde{F}_{\ell+1} + \alpha_{2\ell+1} x_2 \tilde{F}_{\ell+1}),$$

and $F_{\ell+1} \nmid \tilde{F}_{\ell+1}$, we get that

$$F_{\ell+1} \mid (\alpha_{0\ell+1} x_0 + \alpha_{1\ell+1} x_1 + \alpha_{2\ell+1} x_2).$$

This implies that

$$\alpha_{0\ell+1} = \alpha_{1\ell+1} = \alpha_{2\ell+1} = 0.$$

By the same idea as above, one can show that

$$\alpha_{ij} = 0$$

for every $0 \leq i \leq 2$ and $\ell + 1 \leq j \leq s$, and thus

$$\dim_{\mathbb{k}}(I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-1} = 4s - 3\ell.$$

In other words, the Hilbert function of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ in degree $2s - \ell - 1$ is

$$\mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(2s - \ell - 1) = \binom{(2s-\ell-1)+2}{2} - (4s - 3\ell). \quad (4.7)$$

Now we shall show that L is a Lefschetz element of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$. Note that

$$I_{\mathbb{X}} + I_{\mathbb{Y}} = (\tilde{F}_1, \dots, \tilde{F}_s, L \tilde{G}_1, \dots, L \tilde{G}_s, \prod_{i=1}^s G_i).$$

Consider a multiplication map by L

$$\times L : (R/(I_{\mathbb{X}} + I_{\mathbb{Y}}))_{2s-\ell-2} \rightarrow (R/(I_{\mathbb{X}} + I_{\mathbb{Y}}))_{2s-\ell-1} \quad (4.8)$$

and let $N \in \text{Ker}(\times L)$. Then

$$\begin{aligned} N \cdot L &\in (I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-1} \\ &= \langle \tilde{F}_1, \dots, \tilde{F}_\ell, x_0 \tilde{F}_{\ell+1}, x_1 \tilde{F}_{\ell+1}, x_2 \tilde{F}_{\ell+1}, \dots, x_0 \tilde{F}_s, x_1 \tilde{F}_s, x_2 \tilde{F}_s, L \tilde{G}_{\ell+1}, \dots, L \tilde{G}_s \rangle. \end{aligned}$$

This implies that for some $\alpha_1, \dots, \alpha_\ell \in \mathbb{k}$, $K_{\ell+1}, \dots, K_s \in R_1$, and $\beta_{\ell+1}, \dots, \beta_s \in \mathbb{k}$,

$$N \cdot L = \sum_{i=1}^{\ell} \alpha_i \tilde{F}_i + \sum_{i=\ell+1}^s K_i \tilde{F}_i + \sum_{i=\ell+1}^s \beta_i L \tilde{G}_i,$$

i.e.,

$$\left(N - \sum_{i=\ell+1}^s \alpha_i \tilde{G}_i \right) \cdot L = \sum_{i=1}^{\ell} \alpha_i \tilde{F}_i + \sum_{i=\ell+1}^s K_i \tilde{F}_i \in I_{\mathbb{X}}.$$

In other words, $N - \sum_{i=\ell+1}^s \alpha_i \tilde{G}_i$ is contained in the kernel of the multiplication map by L

$$\times L : (R/I_{\mathbb{X}})_{2s-\ell-2} \rightarrow (R/I_{\mathbb{X}})_{2s-\ell-1}.$$

Since L is a general linear form in R_1 , this multiplication map by $\times L$

$$(R/I_{\mathbb{X}})_i \rightarrow (R/I_{\mathbb{X}})_{i+1}$$

is injective for all $i \geq 0$. It follows that

$$N - \sum_{i=\ell+1}^s \alpha_i \tilde{G}_i \in (I_{\mathbb{X}})_{2s-\ell-2},$$

and so

$$N \in (\tilde{F}_{\ell+1}, \dots, \tilde{F}_s, \tilde{G}_{\ell+1}, \dots, \tilde{G}_s)_{2s-\ell-2}.$$

Moreover, since $\tilde{F}_{\ell+1}, \dots, \tilde{F}_s \in (I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-2}$, but $\tilde{G}_{\ell+1}, \dots, \tilde{G}_s \notin (I_{\mathbb{X}} + I_{\mathbb{Y}})_{2s-\ell-2}$, we get that

$$\dim_{\mathbb{k}}[\text{Ker}(\times L)]_{2s-\ell-2} = s - \ell.$$

Hence

$$\begin{aligned} \dim_{\mathbb{k}}[\text{Im}(\times L)]_{2s-\ell-1} &= \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(2s-\ell-2) - \dim_{\mathbb{k}}[\text{Ker}(\times L)]_{2s-\ell-2} \\ &= \left[\binom{2s-\ell-2+2}{2} - (s-\ell) \right] - (s-\ell) \\ &= \binom{(2s-\ell-1)+2}{2} - (2s-\ell) - 2(s-\ell) \\ &= \binom{(2s-\ell-1)+2}{2} - (4s-3\ell) \\ &= \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(2s-\ell-1), \quad (\text{by equation (4.7)}). \end{aligned}$$

This indicates that the multiplication map by L in equation (4.8) is surjective.

Now consider the following exact sequence:

$$0 \rightarrow ((I_{\mathbb{X}} + I_{\mathbb{Y}} : L)/(I_{\mathbb{X}} + I_{\mathbb{Y}}))_{d-1} \rightarrow A_{d-1} \xrightarrow{\times L} A_d \rightarrow (A/LA)_d \rightarrow 0.$$

Since the multiplication map by L is surjective in degree $2s - \ell - 2$, we have

$$\dim_{\mathbb{k}}(A/LA)_d = 0$$

for $d \geq 2s - \ell - 1$. This means the multiplication map by L is surjective in degrees $d \geq 2s - \ell - 2$. Therefore, L is a Lefschetz element of A . This completes the proof. \square

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